

## NOTES ON PRACTICAL HYDROGRAPHY.

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### USE OF THE LAMBERT CONFORMAL CONICAL PROJECTION FOR THE PLOTTING OF HYDROGRAPHIC SURVEYS.

by

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#### I.

We propose to show in the course of this article that the LAMBERT conformal conical projection, *when used to represent a zone not widely extended in latitude and as extensive as desired in longitude*, constitutes an excellent "*plotting sheet projection*" for the plotting of surveys executed for cartographic purposes.

In order to avoid all ambiguity it is necessary to define exactly what is meant by "*plotting sheet projection*".

We designate by this appellation the plane projection which, *once we have plotted on it the positions of all the geodetic points, that is to say, the triangulation points, by means of appropriate methods of calculation*, replaces the surface of the terrestrial spheroid for the determination of all the topographical and hydrographic (soundings) details of the area in question *without prejudice to the projection which will be employed for the published chart.* (1)

As we know, this determination is made either by means of angular measurements between geodetic points, or by means of distances measured from these geodetic points.

It is evident that the substitution of a plane for the curved surface of the earth greatly simplifies the determination, which is obtained by means of several rapid calculations in plane geometry and often by means of simple graphic methods.

In order that the projection may answer the purpose intended *it is evidently necessary that the distances and the bearings measured on the plane projection may be considered as equal to the distances and the bearings observed on the surface of the geoid.* This condition is realised in the LAMBERT projection when it is used for the purpose and with the limitations mentioned.

It may almost be asserted that the excellent qualities of this projection, although not unknown, have been systematically forgotten for some time, especially in hydrography.

To the best of our knowledge, the first application of the LAMBERT projection to the plotting of the topographic and cadastral surveys was made in the Grand Duchy of Mecklenburg towards the end of the last

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(1) In fact it is evident that the projection used for the chart itself frequently does not possess the qualities of a good working projection owing to finite errors in the distances and bearings on the chart. The Mercator projection, for instance, which is an excellent working projection on the equator, is poor in the regions extending to even only slightly higher latitudes.

century (JORDAN - "*Handbuch der Vermessungskunde*" - 7th Edition - 1923 - Part III, pages 346 and 538).

It is aptly stated in Special Publication N° 68 of the Coast and Geodetic Survey (2) that its advantages were evidently not fully appreciated until the beginning of the World War. It appears in fact from this Publication and still more from two other Publications, N° 47 and N° 49, which preceded it and which at present are out of print (3), that the LAMBERT projections were employed by the French Army (under the name of "*plan directeur*") to map the territory comprised between the parallels  $46^{\circ}48'$  and  $52^{\circ}12'$  N. and the meridians  $1^{\circ}48'$  and  $9^{\circ}00'$  East of Paris, a zone extending over  $5^{\circ}24'$  in latitude, *i. e.* about 600 kilometres.

This projection was then (4) adopted by the French *Service Hydrographique* (under the name of *Système Lambert du Nord de l'Algérie*) for the execution of the new survey of the coasts of Algiers and Tunis. This was a particularly happy choice in view of the narrow extent in latitude and the wide expanse of the region in longitude. As far as we know, this is the sole instance to date in which this projection has been applied in the realm of hydrography.

On the other hand it should be noted that, in an article published by A. WEDEMAYER in the *Annalen der Hydrographie* of March-April 1919, page 49 *et seq.* which deals precisely with the question of the choice of the working projection for the plotting of coastal surveys and in which the problem is minutely studied and comparisons drawn between the different types of conformal and non-conformal projections, the LAMBERT projection is not even mentioned. (5)

(2) Elements of Map Projection - by DEETZ and ADAMS - 1921.

(3) N° 47 - The LAMBERT Conformal conic projection, by DEETZ, 1918 (out of print).

N° 49 - The LAMBERT Projection Tables, by DEETZ, 1918 (out of print). The Numbers 47 and 49 have now been actually reassigned to New Special Publications of the U. S. Coast & Geodetic Survey which do not refer to the LAMBERT projection. See also note of Arthur R. HINKS: *On the projection adopted for the Allied Maps on the Western Front*, in the *Geographical Journal* - Vol. LVII N° 6, June 1921.

(4) See *Annales Hydrographiques*, Vol. of 1927-28, M. P. MARTI: *Mission Hydrographique sur les côtes d'Algérie et Tunisie*, page 289.

The *Service Géographique* of the French Army has also recently adopted the LAMBERT projection for construction of the Map of France. (See: *Côtes de Provence et Alpes-Maritimes*).

(5) A comparison of the other systems of projection with the LAMBERT projection would have shown, for instance, that the LAMBERT projection lends itself *better than any other* to the mapping of the German coasts (North Sea and Baltic) by selecting a single origin for the co-ordinates  $x$  and  $y$  of the projection. These coasts are in fact included in a zone whose depth does not greatly exceed  $2^{\circ}$  of latitude.

The qualities of this projection have been much discussed, however, particularly in Germany as is shown by the following passage extracted from JORDAN's work (*loc. cit.* page 540): -With regard to the actual practice followed in selecting a basis for the topographical and land surveys, the triangulation of Mecklenburg with its conformal projection, is the best of all which have been used by the German Topographical Service. In replying to a contradictory assertion on this subject, Engineer VOGELER at Schwerin has shown in a clear and convincing manner in the *Zeitschrift für Vermessungskunde*, 1896 - pp. 257-263, the superiority of the Mecklenburg projection over all other German projections and in particular over the so-called "SOLDNER Projection".

## II.

We should like to be permitted to commence this article with a summary description of the LAMBERT conical conformal projection. The purpose of this is not to call to mind the well-known ideas of those who have studied the theory of geographic projections, which are amply developed in a large number of treatises (6), but rather to clarify the definitions and the properties of which we shall make use in the discussion of the plotting-sheet projection in a given region :-

a) With regard to the *representation of terrestrial meridians*, the LAMBERT projection (like all strictly conical projections), may be considered as obtained

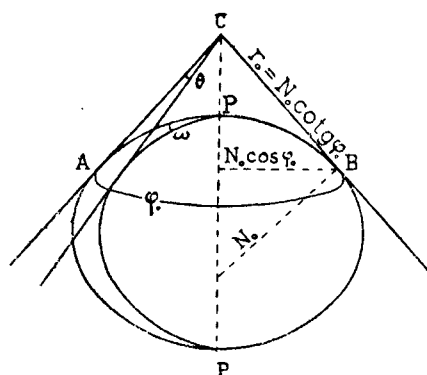


Fig. 1

by the development on a plane of the surface of the cone circumscribed about the terrestrial spheroid, (or, in general, about a reduction of this spheroid, obtained by multiplying its dimensions by a definite factor ; - *scale*) at a given parallel  $AB$  (Fig. 1) of latitude  $\varphi_0$  which we will call the *standard parallel*. The planes of the terrestrial meridians intersect the conical surface in straight lines converging on a point  $C$  at angles to each other proportional to their respective differences in longitude.

The ratio of proportionality  $l$  between the difference in longitude of two meridians and the angle  $\theta$  (convergence of the meridians on the projection) formed by the two straight lines which represent them on the projection, is equal to  $\sin \varphi_0$ .

$$(1) \quad \theta = \omega \sin \varphi_0.$$

b) Each parallel is represented by a circle with the centre located at  $C$  (which represents the pole) and the radius  $r$  given by the formula :-

$$(2) \quad r = K \cot l \left( 45^\circ + \frac{\varphi}{2} \right) \left( \frac{1 + e \sin \varphi}{1 - e \sin \varphi} \right)^{\frac{dl}{2}}$$

in which  $\varphi$  = latitude of the parallel ;  $e$  = eccentricity of the terrestrial meridian  $l \sin \varphi_0$  ;  $K$  = constant, the value of which is obtained in a very simple manner on the hypothesis that the distances, measured along the standard parallel, will be represented in their true lengths. Therefore by definition, the radius  $r$  of the standard parallel will be equal to the side  $CA$  of the cone, circumscribed about the true spheroid *i. e.*

$$(3) \quad \begin{aligned} CA &= \frac{AD}{\sin \varphi_0} = \frac{N_0 \cos \varphi_0}{\sin \varphi_0} \\ r_0 &= N_0 \cot \varphi_0. \end{aligned}$$

(6) Special mention should be made of the Special Publications by Oscar S. ADAMS, Nos 52 and 53, U. S. Coast & Geodetic Survey, 1918, on account of their *essentially practical character*.

in which  $N_0$  is the great normal for the latitude  $\varphi_0$ . Further by formula (2) we should have

$$(3a) \quad r_0 = K \cot^l \left( 45^\circ + \frac{\varphi_0}{2} \right) \left( \frac{1 + e \sin \varphi_0}{1 - e \sin \varphi_0} \right)^{\frac{el}{2}}$$

The parallel  $\varphi_0$  having been fixed, the group of formulae (3) and (3a) determines the value of the constant  $K$  in the case under consideration. (The constant  $K$  is the measure of the radius of the circle representing the equator and is, in fact, the special value of  $r$  (by formula 2) for  $\varphi = 0$ ).

On the projection the pole  $P$  only of the hemisphere to which the standard parallel belongs is represented (we call this the *elevated pole*). The opposite pole  $P'$  (the *lower pole*) is located at infinity.

c) *The modulus of linear dilation* (7) depends solely on the latitude. It is therefore constant everywhere along the same parallel. Thus it is evident that its value is equal to the ratio of the total length of the parallel on the projection ( $l \approx \pi r$ ) to the total length of the parallel taken on the earth ( $2 \pi N \cos \varphi$ ).

That is :-

$$(4) \quad \text{Modulus} = \frac{l r}{N \cos \varphi}$$

( $N$  = the great normal at  $\varphi$ )

This modulus reaches its minimum value on the standard parallel (which, for this reason, is also termed the *parallel of minimum dilation*), and increases steadily on each side of this parallel up to the two poles. In the vicinity of  $\varphi_0$  the variations are extremely slow, that is to say, the same length  $s$  measured on the earth along the standard parallel and along other adjacent parallels is represented on the projection by the lengths  $s'$ ,  $s''$ ,  $s'''$ ..... which differ very slightly from each other.

A *very faithful* representation is thus given of the earth's surface over the entire zone in which this condition holds true. We see immediately from this that the LAMBERT projection is particularly suitable for representing on a plane areas which have but little extent in latitude, whatever their extent in longitude may be, by *making the central parallel of the area in question the standard parallel*.

### III.

In this very brief summary of the theory of the LAMBERT projection, we have stated nothing, we repeat, which has not already been abundantly developed in a large number of treatises on geographical projections. Less known, however, are the considerations which we propose to develop later on the characteristics and the method of construction of the LAMBERT projection considered as a "*plotting sheet*" projection.

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(7) That is, the ratio  $\frac{ds_1}{ds}$  of the infinitely small linear element  $ds_1$  issuing from a given point of the projection and the corresponding element  $ds$  of the terrestrial surface.

We shall follow the method first set out by P. PIZZETTI in his *Trattato di Geodesia Teoretica* (1st Edition 1905 - Chapter XV.).

Having selected as the standard parallel the central parallel of the area to be represented and having agreed that the distances measured along this parallel will be represented in their true lengths, the radius  $r_o$  of the parallel  $\varphi_o$  of the projection is determined by formula (3) ( $r_o = N_o \cotg \varphi_o$ ). We may then determine the radii  $r$  of the other parallels by formula (4), but since we are dealing with parallels very close to the standard parallel it is more convenient, for numerous reasons, to determine ( $r_o - r$ ) of the arc of the meridian on the projection by a development in series of the same arc on the earth as a function of the length  $\beta$ , (8).  $\beta$  thus being the arc of the terrestrial meridian comprised between the given parallel  $\varphi$  and the standard parallel, PIZZETTI obtains the following expression by neglecting the terms of the order  $\beta^5$  and the terms  $e^2 \beta^4$ :-

$$(5) \quad r_o - r = \beta \left\{ 1 + \frac{\beta^2}{6 \rho_o N_o} + \frac{\beta^3}{24} \frac{\tan \varphi_o}{N_o^2 \rho_o} \right\}$$

( $\rho_o$  = the radius of curvature of the meridian at latitude  $\varphi_o$ ) (9). Where the value of the difference  $\varphi - \varphi_o$  is small, the last term of the formula, i. e. the quantity:-

$$\frac{\beta^4}{24} \frac{\beta \tan \varphi_o}{N_o^2 \rho_o},$$

may be neglected. Its value is in fact, roughly equal to (10):

$$\frac{(\Delta^\circ \varphi)^4}{40} \tan \varphi_o, \text{ metres}$$

in which  $\Delta^\circ \varphi$  represents the difference  $\varphi - \varphi_o$  expressed in degrees. We have therefore for  $\varphi_o = 45^\circ$ : 2.5 centimetres, for  $\Delta^\circ \varphi = 1^\circ$ ; 12 centimetres for  $\Delta^\circ \varphi = 1.5^\circ$ ; 40 centimetres for  $\Delta^\circ \varphi = 2^\circ$ ; one metre for  $\Delta^\circ \varphi = 2.5^\circ$ .

According to the accuracy desired this may or may not be taken into account.

It is evident that it will often be permissible to apply the simplified formula:-

$$(5a) \quad r_o - r = \beta + \frac{\beta^3}{6 \rho_o N_o}$$

(8) JORDAN in his *Handbuch der Vermessungskunde* considers the development of formula (2) in terms of latitude difference, but it appears more convenient to carry out this development in terms of  $\beta$ .

(9) The development of PIZZETTI is given in full with a large number of scientific discussions in a Special Publication of the *Royal Geographical Society* of London, which is well worth the attention of those interested in the theory of geographical projections:- A. E. YOUNG "Some Investigations in the Theory of Map Projections" - London 1920. Chapter III, page 57 et seq. are specially devoted to the LAMBERT projection.

See also for this development page 13 of Special Publication N° 47 of the U. S. Coast & Geodetic Survey previously cited, Note (3).

(10) On the hypothesis that the earth is spherical this term becomes.

$$\frac{R \text{ arc}^4 \Delta \varphi}{24} \tan \varphi_o$$

( $R$  = terrestrial radius) and, expressing  $R$  in metres,

$$\frac{R \text{ arc } 1^\circ}{24} = \frac{1}{40}, \text{ about}$$

In the applications of this projection which have been made up to the present by the French Army and by the French Hydrographic Service (*see* para. 1), this simplification has always been admitted, and further, in formula (5 a)  $N_o^2$  has been substituted for  $\rho_o N_o$ . In other words the following formula has been used :-

$$(5 b) \quad r_o - r = \beta + \frac{\beta^3}{6 N_o^2}$$

We now have all the elements necessary to determine the cartesian orthogonal co-ordinates  $x, y$ , of the points of the plane projection in terms of the corresponding geographical co-ordinates of the terrestrial points

On the plane of Figure 2, let us take the meridian  $OP$  as the axis of  $x$  (positive towards  $P$ ) selected as the first meridian. Let us take  $O$  in latitude  $\varphi_o$  as the origin of co-ordinates so that the axis of  $y$  will be tangent to the parallel  $\varphi_o$ . Consequently a point  $A$  with the polar co-ordinates  $r$  and  $\theta$  will have the following cartesian co-ordinates :-

$$x = r_o - r \cos \theta; \quad y = r \sin \theta$$

or again :-

$$(6) \quad \begin{cases} x = 2 r_o \sin^2 \frac{\theta}{2} + (r_o - r) \cos \theta \\ y = r_o \sin \theta - (r_o - r) \sin \theta \end{cases}$$

The geographical co-ordinates  $\varphi$  and  $\omega$  of the position under consideration being given, we calculate the arc  $\beta$  by the formula

$$(7) \quad \beta = (\varphi - \varphi_o)'' \rho_m \text{ arc } 1''$$

in which  $\rho_m$  represents the radius of curvature of the meridian for the latitude  $\frac{\varphi + \varphi_o}{2}$  (middle latitude) (11). We obtain the convergence  $\theta$  by formula (1) which we repeat here for convenience :-

$$(8) \quad \theta = \omega \sin \varphi_o.$$

$\beta$  and  $\theta$  being known from formulae (3), (5) and (6) are used to calculate  $x$  and  $y$ .

(11) It is evident that this formula is approximate only, but the approximation is certainly close enough when the difference  $\varphi_o - \varphi = \Delta \varphi$  is small: *e. g.* below  $3^\circ$ .

In fact the error introduced in using this formula to determine  $\beta$  is practically equal in absolute value to

$$E = 0,0285 \cos 2 \varphi_m (\Delta^\circ \varphi)^3, \text{ metres}$$

in other words in every case

$$E < 0,0285 (\Delta^\circ \varphi) \text{ metres.}$$

In this formula  $\Delta^\circ \varphi$  expresses  $\Delta \varphi$  in degrees.

Where  $\Delta \varphi = 2^\circ$ , then  $E < 0,23$  metres, (approx.)

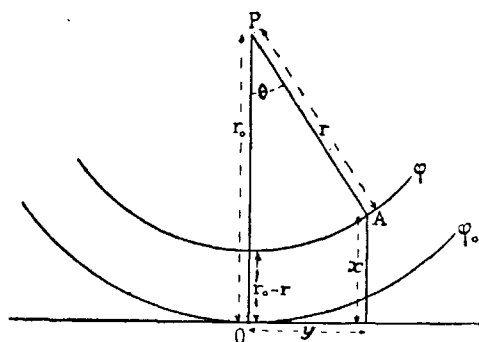


Fig. 2

The calculations are greatly shortened when the geodetic positions ( $\varphi, \omega$ ) which serve as points of departure in the determinations of the points on the projection, have been calculated on one of the ellipsoids of reference (BESSEL, CLARKE, Madrid 1924, etc.) for which tables are published giving the direct values (or better the logarithms) of the quantities  $\rho, N, \sqrt{\rho N}$

*Remark.* — PIZZETTI also gives the formulae which permit the inverse transformation to be made, *i. e.* to calculate  $\varphi$  and  $\omega$  when the plane co-ordinates  $x$  and  $y$  are known. Use is then made of the following formulae:

$$\tan \theta = \frac{y}{r_0 - x}$$

$$r_0 - r = \left( x - 2 r_0 \sin^2 \frac{\theta}{2} \right) \frac{1}{\cos \theta}$$

$$(*) \quad \beta = (r_0 - r) \left\{ 1 - \frac{(r_0 - r)^2}{6 \rho_0 N_0} - \frac{(r_0 - r)^3 \tan \varphi_0}{24 N_0^2 \rho_0} \right\}$$

Having obtained  $\theta$  and  $\beta$ , formulae (7) and (8) give  $\varphi$  and  $\omega$  respectively.

Frequently the term  $(r_0 - r)^4$  may be neglected in formula (\*). The neglect of this term is justified when it is a question of the determination of points of detail and in this case we may use the simplified formula:-

$$(**) \quad \beta = (r_0 - r) \left\{ 1 - \frac{(r_0 - r)^2}{6 \rho_0 N_0} \right\}$$

#### IV.

On the plane projection thus obtained, *i. e.* constructed with the co-ordinates  $x, y$ , which have just been determined, *the lengths are unchanged on the standard parallel*, a fact which it is well to remember. For every other latitude  $\varphi$ , there is a dilation defined, as we have shown above, by the value on that parallel of the modulus of the linear dilation.

According to the general definition (See note 7) the modulus is equal to the ratio  $\frac{ds_1}{ds}$  of the infinitely small element  $ds_1$  (taken on the projection) to the corresponding element  $ds$  (taken on the surface of the earth). Let us determine its value by considering the infinitely small element of the meridian at the parallel under consideration. We then have  $\text{modulus} = \frac{d(r_0 - r)}{d\beta}$  in which  $(r_0 - r)$  and  $\beta$  are the lengths of the arc of the meridian comprised between  $\varphi_0$  and  $\varphi$  on the Earth and on the projection respectively.

By formula (5a) which is a sufficiently close approximation for this determination, we have:-

$$d(r_0 - r) = d\beta \left( 1 + \frac{\beta^2}{2 \rho_0 N_0} \right)$$

and consequently,

$$\text{modulus} = 1 + \frac{\beta^2}{2 \rho_o N_o}$$

We can further simplify this formula by noting that according to formula (7) :-

$$\beta^2 = \rho_m^2 \text{arc}^2 (\varphi - \varphi_o)$$

and that very closely :-

$$\rho_m^2 = \rho_o N_o$$

We obtain therefore :-

$$(9) \quad \text{modulus} = 1 + \frac{1}{2} \text{arc}^2 (\varphi - \varphi_o)$$

or again, the value of  $(\varphi - \varphi_o)$  being small,

$$(9a) \quad \text{modulus} = \frac{1}{\cos (\varphi - \varphi_o)}$$

The formulae (9) and (9a) are very simple and may advantageously replace the exact formula (4) in the following discussion.

It follows from these formulae that the linear dilation depends on the distance from the standard parallel and not on the absolute values of  $\varphi$  or of  $\varphi_o$ . On the parallels equidistant from  $\varphi_o$  and on each side of this parallel there will be equal linear deformations.

We may say then that a certain finite length  $s$  taken along the *standard parallel* on the surface of the earth, is faithfully represented on the projection and further, that the *same* length taken along another terrestrial parallel  $\varphi$  becomes  $s'$  on the projection

$$s' = s \frac{1}{\cos (\varphi - \varphi_o)}$$

From this formula we deduce :-

$$s' - s = s \left( \frac{1}{\cos (\varphi - \varphi_o)} - 1 \right)$$

or, further :-

$$(10) \quad s' - s = s \frac{\text{arc}^2 (\varphi - \varphi_o)}{2}$$

The difference  $s' - s$  is a measure of the *dilation* caused by the projection of the corresponding length  $s$ , measured on the earth's surface. From formula (10) we have :-

TABLE 1.	{	for $\varphi - \varphi_o = 1^\circ$	the dilation =	0.15	metre	per	kilometre.
		" " " $1^\circ 30'$	"	0.34	"	"	"
		" " " $2^\circ$	"	0.61	"	"	"
		" " " $2^\circ 30'$	"	0.95	"	"	"
		" " " $3^\circ$	"	1.37	"	"	"

In view of the results deduced from this, it is interesting to determine at what distance from the standard parallel the dilation will become equal



to half the maximum dilation corresponding to the extreme latitudes  $\varphi_x, \varphi'_x$  of the projection (Fig. 3.)

Let  $\varphi_1$  and  $\varphi'_1$  be the two parallels sought, and let us postulate:-

$$\Delta \varphi_x = \varphi_x - \varphi_0 = \varphi_0 - \varphi'_x$$

$$\Delta \varphi_1 = \varphi_1 - \varphi_0 = \varphi_0 - \varphi'_1$$

From formula (10)

$$\text{maximum dilation} = \frac{\text{arc}^2 \Delta \varphi_x \cdot s}{2}$$

$$\text{dilation in } \varphi_0 = \frac{\text{arc}^2 \Delta \varphi_1 \cdot s}{2}$$

By hypothesis:-

$$\text{dilation at } \varphi_1 = \frac{\text{max. dilation}}{2}$$

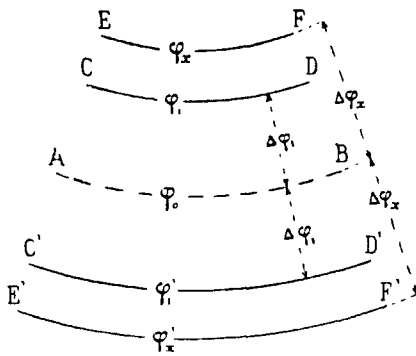


Fig. 3

$$\text{And for this: } \text{arc}^2 \Delta \varphi_1 = \frac{1}{2} \text{arc}^2 \Delta \varphi_x$$

that is:-

$$(11) \quad \Delta \varphi_1 = \sqrt{\frac{1}{2}} \cdot \Delta \varphi_x$$

which formula solves the problem in a simple manner.

## V.

This being granted let us subject the projections obtained to a *slight linear contraction* in such a manner that the *linear dilation* becomes zero for the latitudes  $\varphi_1, \varphi'_1$  which we have just determined. To obtain this result it is necessary to multiply the co-ordinates  $x, y$  already calculated by the inverse of the modulus at the latitude  $\varphi_1$ , given by formula (11) i. e. by the coefficient:-

$$c = \cos (\varphi_1 - \varphi_0)$$

or

$$c = 1 - \text{versine } (\varphi_1 - \varphi_0)$$

$$X = c.x \quad Y = c.y$$

Then, in the *reduced projection*, that is, in the projection constructed with the new values  $XY$  of the plane co-ordinates, the lengths will be represented in their true magnitudes along the parallels  $\varphi_1, \varphi'_1$  whereas:-

a) In the zone  $CD C'D'$  comprised between these two parallels and divided in the middle by  $\varphi_0$ , the lengths on the projection will be less than the true lengths, that is to say, they will be subjected to a *contraction* with respect to the corresponding lengths measured on the earth and this contraction will reach a maximum on the standard parallel.

b) In the extreme zones  $E F C D, E' F' C' D'$ , the lengths on the projection will be greater than the true lengths, and the maximum dilations will be on the limiting parallels  $\varphi_x$  and  $\varphi'_x$ .

It is readily seen that the contractions and the maximum dilations have the same absolute values, and that this value is equal to half of the maximum dilation which existed on the limiting parallels  $\varphi_x$  and  $\varphi'_x$  on the *original projection* (constructed with  $x$  and  $y$ ).

We obtain therefore (See Table I):

TABLE II.

$\Delta \varphi_x$	Total height in Latitude of the region = $2 \Delta \varphi_x$	Coefficient $c$	Maximum linear alteration in the reduced projection.
$1^\circ$	$2^\circ$ or 220 km. (about)	$1 - 0.000076$	0.08 metre per km.
$1^\circ 30'$	$3^\circ$ » 330 » »	$1 - 0.000171$	0.17 » »
$2^\circ$	$4^\circ$ » 440 » »	$1 - 0.000305$	0.30 » »
$2^\circ 30'$	$5^\circ$ » 550 » »	$1 - 0.000476$	0.48 » »
$3^\circ$	$6^\circ$ » 660 » »	$1 - 0.000685$	0.68 » »

Thus it is advantageous to replace the *original projection* by the *reduced projection*, i. e. established with the values  $X$  and  $Y$ .

*Remark.* — From what has already been said in the remark under Para. III, it has been shown that if we wish to change from the co-ordinates  $XY$  of the reduced projection to the geographical co-ordinates  $\varphi$  and  $\omega$ , it is first necessary to obtain from  $XY$  the corresponding values  $x, y$ , of the original projection :-

$$x = \frac{X}{c}; \quad y = \frac{Y}{c}$$

and then to apply the transformation formulae (\*) and (\*\*).

## VI.

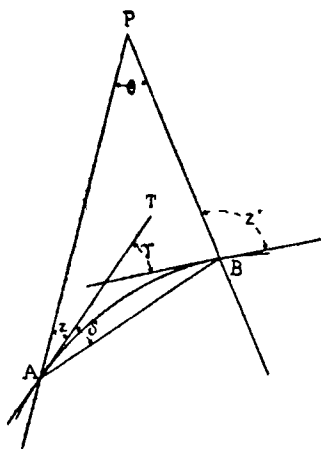


Fig. 4

The term *bearing error* will be applied to the difference between the value of the observed bearing of a position  $B$  from  $A$ , taken in the plane of the projection and the value of the same bearing taken on the surface of the earth. The geodetic which joins these two points is represented (Fig. 4) by a curved line and, since the projection is conformal, the bearing of point  $B$  from  $A$ , which is equal to  $PAT$  on the surface of the earth ( $AT$  being the tangent to the curve at  $A$ ) is represented on the projection by the angle  $PAB$ . The angle  $TAB = \delta$  between the tangent  $AT$  and the chord  $AB$  is the amount of the bearing error. In order that it may be permissible to substitute the plane projection for the earth's surface, it is necessary that the

angle  $\delta$  should be so small that it may be *neglected* in topographic operations. We shall determine this error, if not strictly accurately (12), at least with an approximation sufficient to show its magnitude. In our demonstration we assume that within the contours of the area represented the earth is spherical. This hypothesis is perfectly admissible as the area considered is not extensive and under these conditions the geodetics become arcs of a great circle (orthodromes).

To evaluate the angle  $\delta = TAB$ , note that if the positions  $A, B$ , are close together, we may assume that the arc of the curve  $AMB$  coincides with the arc of the circle of which the tangents at the extremities  $A$  and  $B$  have the same contingency angle  $\gamma$  as the tangents to the curve. It is readily seen from the figure that this angle is equal to the difference between the variation  $\alpha = Z' - Z$  which the azimuth undergoes along the orthodrome in passing from  $A$  to  $B$ , and the convergence  $\theta = APB$  of the meridians of  $A$  and  $B$ . Seeing that approximately (*i. e.* by neglecting the terms of the terms of the higher order) (13) :

$$Z' - Z = \Delta \omega \sin \varphi_m$$

( $\Delta \omega$  represents the difference in longitude between  $A$  and  $B$  ;  $\varphi_m$  is the middle latitude between  $A$  and  $B$ ) and, further, that :-

$$\theta = \Delta \omega \sin \varphi_o$$

the contingency angle  $\gamma$  is given by the equation :-

$$\gamma = \Delta \omega (\sin \varphi_m - \sin \varphi_o)$$

And finally we have :-

$$TAB = \delta = \frac{1}{2} \gamma = \frac{\Delta \omega}{2} (\sin \varphi_m - \sin \varphi_o).$$

With the aid of this formula we see that in the areas adjacent to the standard parallel (and certainly between the limits  $\Delta \varphi_x$  under consideration), the error  $\delta$  is of a magnitude compatible with the accuracy required for the determination of the points of detail by means of angular measurements.

*Remark.* — By following a line of reasoning analogous to the above, it may be proved that the geodetics are curves with their concavity always turned towards the standard parallel, that the geodetics which intersect this parallel have a *point of inflection* and that, consequently, the geodetics from the parallel  $\varphi_o$  practically coincide with straight lines for some distance, etc...



(12) See Chapter V of the work cited in Note 9.

(13) The exact formula gives :-

$$Z' - Z = \Delta \omega \sin \varphi_m \left( 1 + \frac{\text{arc}^2 \Delta \varphi}{8} + \frac{\text{arc}^2 \Delta \omega}{12} \cos^2 \varphi_m \right)$$